

ON THE FIRST GAUSSIAN MAP FOR PRYM-CANONICAL LINE BUNDLES

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ABSTRACT. We prove by degeneration to Prym-canonical binary curves that the first Gaussian map μ_A of the Prym canonical line bundle $\omega_C \otimes A$ is surjective for the general point $[C, A] \in \mathcal{R}_g$ if $g \geq 12$, while it is injective if $g \leq 11$.

1. INTRODUCTION

Consider the moduli space \mathcal{R}_g parametrizing isomorphism classes of pairs $[C, A]$, where C is a smooth projective curve of genus g and A is a non trivial two torsion line bundle on C . In this paper we prove that for a general point $[C, A]$ in \mathcal{R}_g , the first Gaussian map of the Prym-canonical line bundle $\omega_C \otimes A$ is of maximal rank, i.e. it is surjective for $g \geq 12$ and injective for $g \leq 11$.

We do it by degeneration to certain stable binary curves of genus g embedded in \mathbb{P}^{g-2} by a Prym-canonical linear system $|\omega_C \otimes A|$ that have been constructed in [9] to prove the surjectivity of the second Gaussian map of such line bundles for the general element in \mathcal{R}_g for $g \geq 20$. A stable binary curve of genus g is the union of two smooth rational curves meeting transversally at $g + 1$ distinct points and in [5] Calabri, Ciliberto and Miranda have first used these curves to prove by degeneration the maximality of the rank of the second Gaussian map of the canonical line bundle for a general curve.

The first Gaussian (or Wahl) map of the canonical line bundle has been extensively studied by many authors. Here we only recall that Wahl's theorem ([14], see also [4]) says that if C is a curve sitting on a K3 surface, then its first Wahl map is not surjective, while Ciliberto, Harris and Miranda have proved in [7] that the first Wahl map is surjective for the general curve of genus $g \geq 10$, $g \neq 11$ (see also [13]).

Here we focus on the study of the first Gaussian map $\mu_A : \Lambda^2 H^0(C, \omega_C \otimes A) \rightarrow H^0(C, \omega_C^{\otimes 3})$, where μ_A is the Gaussian map $\gamma_{\omega_C \otimes A, \omega_C \otimes A}^1$ defined in section 3.1. We also point out that in [8] it is shown by degeneration to graph curves that the first Gaussian map $\gamma_{\omega_C, \omega_C \otimes A}^1$ is surjective for the general point in \mathcal{R}_g for $g \geq 12$, $g \neq 13, 19$.

The main result of this paper says that the first Gaussian map μ_A of a general Prym-canonical binary curve of genus g is surjective for $g \geq 12$. In the case $g = 12$ we verify the surjectivity of μ_A on explicit Prym-canonical binary curves using Maple and similar computations with Maple also give the injectivity of μ_A for a general Prym-canonical binary curve of genus $g \leq 11$. Then, to prove surjectivity for higher genus g we proceed by induction on g in a similar way as it has been done in [5] and in [9]. Maximality of the rank for the general element in \mathcal{R}_g follows then by semicontinuity.

This implies that the locus of points $[C, A]$ in \mathcal{R}_g for which the map μ_A is not surjective is a proper subscheme of \mathcal{R}_g and in the case $g = 12$ it turns out to be an effective divisor in \mathcal{R}_{12} of which we compute the cohomology class both in \mathcal{R}_{12} and in a partial compactification $\tilde{\mathcal{R}}_{12}$ introduced in [12].

The paper is organized as follows: in section 2 we describe the construction given in [9] of Prym-canonical binary curves. In section 3 we recall the definition of Gaussian maps and we describe the first Gaussian map μ_A for Prym-canonical line bundles both for smooth curves and for Prym-canonical binary curves. In section 4 we prove by induction on the genus the surjectivity of the map μ_A . In section 5 we describe the Maple script that gives the surjectivity of μ_A for $g = 12$ and the injectivity statement for lower genus. In section 6 we compute the cohomology class in $\tilde{\mathcal{R}}_{12}$ of the degeneracy locus of μ_A .

¹ Partially supported by PRIN 2009: "Moduli, strutture geometriche e loro applicazioni" and by INdAM (GNSAGA).
AMS Subject classification: 14H10, 14H45, 13D06.

2. PRYM-CANONICAL BINARY CURVES

We first recall an explicit construction of binary curves embedded in \mathbb{P}^{g-1} by a Prym-canonical linear system $|\omega_C \otimes A|$ where $A^2 \cong \mathcal{O}_C$, A non trivial, which was given in [9]. Let C be a binary curve of genus g , and $A \in \text{Pic}^0(C)$ a nontrivial line bundle. Then $H^0(C, \omega_C \otimes A)$ has dimension $g-1$ and the restriction of $\omega_C \otimes A$ to the component C_j is $\omega_{C_j}(D_j)$ where D_j is the divisor of nodes on C_j . Since $\omega_{C_j}(D_j) \cong \mathcal{O}_{\mathbb{P}^1}(g-1)$, the components are embedded by a linear subsystem of $\mathcal{O}_{\mathbb{P}^1}(g-1)$, hence they are projections from a point of rational normal curves in \mathbb{P}^{g-1} . Viceversa, let us take two rational curves embedded in \mathbb{P}^{g-2} by non complete linear systems of degree $g-1$ intersecting transversally at $g+1$ points. Then their union C is a binary curve of genus g embedded either by a linear subsystem of ω_C or by a complete linear system $|\omega_C \otimes A|$, where $A \in \text{Pic}^0(C)$ is nontrivial (see e.g. [6], Lemma 10). Let us now recall the construction given in [9] (Lemma 3.1) of a binary curve C embedded in \mathbb{P}^{g-2} by a linear system $|\omega_C \otimes A|$ with $A^{\otimes 2} \cong \mathcal{O}_C$, and A non trivial, and let us denote a binary curve with this embedding a Prym-canonical binary curve.

Assume that the first $g-1$ nodes, are $P_i = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 at the i -th place, $i = 1, \dots, g-1$, the remaining two nodes are $P_g := [t_1, \dots, t_{g-1}]$ with $t_i = 0$ for $i = 1, \dots, [\frac{g}{2}]$, $t_i = 1$, for $i = [\frac{g}{2}] + 1, \dots, g-1$. and $P_{g+1} := [s_1, \dots, s_{g-1}]$ with $s_i = 1$ for $i = 1, \dots, [\frac{g}{2}]$, $s_i = 0$, for $i = [\frac{g}{2}] + 1, \dots, g-1$.

Then the component C_j is the image of the map

$$(1) \quad \begin{aligned} \phi_j : \mathbb{P}^1 &\rightarrow \mathbb{P}^{g-2}, \quad j = 1, 2, \text{ where} \\ \phi_1(t, u) &:= \left[\frac{tM_1(t, u)}{(t - a_{1,1}u)}, \dots, \frac{tM_1(t, u)}{(t - a_{k,1}u)}, \frac{-M_1(t, u)d_1a_{k+1,1}u}{A_1(t - a_{k+1,1}u)}, \dots, \frac{-M_1(t, u)d_1a_{g-1,1}u}{A_1(t - a_{g-1,1}u)} \right] \\ \phi_2(t, u) &:= \left[\frac{tM_2(t, u)}{(t - a_{1,2}u)}, \dots, \frac{tM_2(t, u)}{(t - a_{k,2}u)}, \frac{-M_2(t, u)d_2a_{k+1,2}u}{A_2(t - a_{k+1,2}u)}, \dots, \frac{-M_2(t, u)d_2a_{g-1,2}u}{A_2(t - a_{g-1,2}u)} \right] \end{aligned}$$

with $k := [\frac{g}{2}]$, $M_j(t, u) := \prod_{r=1}^{g-1} (t - a_{r,j}u)$, and $A_j = \prod_{i=1}^{g-1} a_{i,j}$, $j = 1, 2$, d_2 is a nonzero constant and $d_1 = \frac{-d_2 A_1}{A_2}$. Notice that we have $\phi_j([a_{l,j}, 1]) = P_l$, $l = 1, \dots, g-1$, $\phi_j([0, 1]) = P_g$, $\phi_j([1, 0]) = P_{g+1}$, $j = 1, 2$. In Lemma 3.1 of [9] it is proven that for a general choice of $a_{i,j}$'s, $C = C_1 \cup C_2$ is a binary curve embedded in \mathbb{P}^{g-2} by a linear system $|\omega_C \otimes A|$ with $A^{\otimes 2} \cong \mathcal{O}_C$ and A nontrivial. In fact, recall that $\text{Pic}^0(C) \cong \mathbb{C}^{*g} \cong \mathbb{C}^{*g+1}/\mathbb{C}^*$, where \mathbb{C}^* acts diagonally, and in Lemma 3.1 of [9] it is shown that our line bundle A corresponds to the element $[(h_1, \dots, h_{g+1})] \in \mathbb{C}^{*g+1}/\mathbb{C}^*$, where $h_i = 1$, for $i < [\frac{g}{2}] + 1$, $h_i = -1$, for $i = [\frac{g}{2}] + 1, \dots, g-1$, $h_g = -1$, $h_{g+1} = 1$, so in particular A is of 2-torsion.

3. THE FIRST PRYM-CANONICAL GAUSSIAN MAP

3.1. Gaussian maps. Let Y be a smooth complex projective variety and let $\Delta_Y \subset Y \times Y$ be the diagonal. Let L and M be line bundles on Y . For a non-negative integer k , the k -th Gaussian map associated to these data is the restriction to diagonal map

$$(2) \quad \gamma_{L,M}^k : H^0(Y \times Y, I_{\Delta_Y}^k \otimes L \boxtimes M) \rightarrow H^0(Y, I_{\Delta_Y|\Delta_Y}^k \otimes L \otimes M) \cong H^0(Y, S^k \Omega_Y^1 \otimes L \otimes M).$$

The exact sequence

$$(3) \quad 0 \rightarrow I_{\Delta_Y}^{k+1} \rightarrow I_{\Delta_Y}^k \rightarrow S^k \Omega_Y^1 \rightarrow 0,$$

(where $S^k \Omega_Y^1$ is identified to its image via the diagonal map), twisted by $L \boxtimes M$, shows that the domain of the k -th Gaussian map is the kernel of the previous one:

$$\gamma_{L,M}^k : \ker \gamma_{L,M}^{k-1} \rightarrow H^0(S^k \Omega_Y^1 \otimes L \otimes M).$$

In this paper, we will exclusively deal with the first Gaussian map for curves C , assuming also that $L = M$.

The map γ_L^0 is the multiplication map of global sections

$$H^0(C, L) \otimes H^0(C, L) \rightarrow H^0(C, L^{\otimes 2})$$

which obviously vanishes identically on $\wedge^2 H^0(L)$. Consequently, $H^0(C \times C, I_{\Delta_C} \otimes L \boxtimes L)$ decomposes as $\wedge^2 H^0(L) \oplus I_2(L)$, where $I_2(L)$ is the kernel of $S^2 H^0(C, L) \rightarrow H^0(C, L^{\otimes 2})$. Since γ_L^1 vanishes on

symmetric tensors, one writes

$$\gamma_L^1 : \wedge^2 H^0(L) \rightarrow H^0(\Omega_C^1 \otimes L^{\otimes 2}).$$

Assume now that the line bundle L is $\omega_C \otimes A$, with $A \in \text{Pic}^0(C)[2]$, and denote by

$$(4) \quad \mu_A := \gamma_{\omega_C \otimes A}^1 : \wedge^2 H^0(\omega_C \otimes A) \rightarrow H^0(\omega_C^{\otimes 3})$$

the first Gaussian map.

It is useful to recall also the local definition of μ_A . Given two sections $\sigma_i \in H^0(\omega_C \otimes A)$, $i = 1, 2$, assume that locally $\sigma_i = f_i(z)dz \otimes l$, where l is a local generator of A . The local expression of the first Gaussian map $\mu_A : \wedge^2 H^0(\omega_C \otimes A) \rightarrow H^0(\omega_C^{\otimes 3})$, is $\mu_A(\sigma_1 \wedge \sigma_2) := (f_1 f_2' - f_2 f_1')(dz)^3$.

Denote by \mathcal{R}_g^0 the open subset of \mathcal{R}_g where there exists the universal family $f : \mathcal{X} \rightarrow \mathcal{R}_g^0$. If $b \in \mathcal{R}_g^0$, we have $f^{-1}(b) = [C_b, A_b]$ where C_b is a smooth irreducible curve of genus g and $A_b \in \text{Pic}^0(C_b)[2]$ is a line bundle of order 2 on C_b . Denote by $\mathcal{P} \in \text{Pic}(\mathcal{X})$ the corresponding Prym bundle and consider the Hodge bundle $f_*\omega_f$, where ω_f is the relative dualizing sheaf of f .

The maps μ_A glue together to give a map of vector bundles on \mathcal{R}_g^0 ,

$$(5) \quad \mu : \wedge^2(f_*(\omega_f \otimes \mathcal{P})) \rightarrow f_*((\omega_f \otimes \mathcal{P})^{\otimes 2} \otimes \omega_f) \cong f_*(\omega_f^{\otimes 3})$$

3.2. Strategy of the proof of surjectivity. In the next sections we will prove the surjectivity of the first Prym-canonical Gaussian map μ_A for the general point $[C, A] \in \mathcal{R}_g$. We will do it by degeneration to binary curves following the method used in [5] for the second Gaussian map of the canonical line bundle and also in [9] for the second Gaussian map of Prym-canonical line bundles. Recall that \mathcal{R}_g admits a suitable compactification $\overline{\mathcal{R}}_g$, which is isomorphic to the coarse moduli space of the stack \mathbf{R}_g of Beauville admissible double covers ([3], [1]) and to the coarse moduli space of the stack of Prym curves ([2]).

Consider the partial compactification $\tilde{\mathcal{R}}_g$ of \mathcal{R}_g introduced in [12]. It is defined as the preimage through the forgetful map $\overline{\mathcal{R}}_g \rightarrow \overline{\mathcal{M}}_g$ of the open subvariety of $\overline{\mathcal{M}}_g$ consisting of 1-nodal irreducible stable curves of genus g . Denote by $\psi : \mathcal{X} \rightarrow \tilde{\mathcal{R}}_g$ the universal family and by $\mathcal{P} \in \text{Pic}(\mathcal{X})$ the corresponding Prym bundle as in [12] 1.1. The map of vector bundles over \mathcal{R}_g^0 , $\mu : \wedge^2(f_*(\omega_f \otimes \mathcal{P})) \rightarrow f_*((\omega_f \otimes \mathcal{P})^{\otimes 2} \otimes \omega_f) \cong f_*(\omega_f^{\otimes 3})$ defined in (5), extends to a map

$$(6) \quad \tilde{\mu} : \wedge^2(\psi_*(\omega_\psi \otimes \mathcal{P})) \rightarrow \psi_*((\omega_\psi \otimes \mathcal{P})^{\otimes 2} \otimes \Omega_\psi^1) \cong \psi_*(\omega_\psi^{\otimes 3} \otimes \mathcal{P}^{\otimes 2} \otimes \mathcal{I}_Z),$$

where $Z = \text{Sing}(\psi)$, $\Omega_\psi^1 \cong \omega_\psi \otimes \mathcal{I}_Z$.

If $[C, A]$ is a point in $\tilde{\mathcal{R}}_g$, the local expression of

$$(7) \quad \mu_A : \wedge^2 H^0(\omega_C \otimes A) \rightarrow H^0((\omega_C \otimes A)^{\otimes 2} \otimes \Omega_C^1)$$

is as follows. In local coordinates let $\sigma_i = f_i(z)\xi \otimes l$, $i = 1, 2$ be two sections of $\omega_C \otimes A$ where ξ and l are local generators of the line bundles ω_C , respectively A , and define

$$(8) \quad \mu_A(\sigma_1 \wedge \sigma_2) := (f_1 df_2 - f_2 df_1)\xi^2 \otimes l^2.$$

If $[C, A]$ is a Prym-canonical binary curve, the map $\mu_A : \wedge^2 H^0(\omega_C \otimes A) \rightarrow H^0(\omega_C^{\otimes 2} \otimes \Omega_C^1)$ can be defined as in (8) and we will give an explicit description of it for the Prym-canonical binary curves constructed in section 2.1.

To prove by semicontinuity the surjectivity of μ_A for the general point in \mathcal{R}_g in the following we will exhibit Prym-canonical binary curves $[C, A]$ for which μ_A is surjective.

3.3. 1st Gaussian map for Prym-canonical binary curves. Assume now that $[C, A]$ is a stable Prym-canonical binary curve as in section 2.1. The map $\mu_A : \wedge^2 H^0(\omega_C \otimes A) \rightarrow H^0(\omega_C^{\otimes 2} \otimes \Omega_C^1)$ is defined as in (8) and in the following we will give an explicit description of it. To this end, let us first describe the space $H^0(\Omega_C^1 \otimes \omega_C^{\otimes 2})$ for a Prym-canonical binary curve $C = C_1 \cup C_2$. If we denote by $\pi : \tilde{C} \rightarrow C$ the normalization map, we have an exact sequence

$$0 \rightarrow \mathcal{T} \rightarrow \Omega_C^1 \rightarrow \mathcal{F}_C \rightarrow 0,$$

where $\mathcal{F}_C \cong \bigoplus_{i=1,2} \pi_* \omega_{C_i}$, \mathcal{T} is a torsion sheaf supported at the nodes of C and if p is a node and the local equation of C around p is $xy = 0$, \mathcal{T}_p is a one dimensional vector space spanned by ydx . So, tensoring with $\omega_C^{\otimes 2}$ and taking global sections, we get an exact sequence

$$(9) \quad 0 \rightarrow T \rightarrow H^0(\Omega_C^1 \otimes \omega_C^{\otimes 2}) \rightarrow \bigoplus_{j=1,2} H^0(C_j, \omega_{C_j}^{\otimes 3}(2D_j)) \rightarrow 0,$$

where, as above, D_j is the divisor of nodes on C_j and $T = H^0(C, \mathcal{T} \otimes \omega_C^{\otimes 2}) \cong \mathbb{C}^{g+1}$.

Consider first the non torsion part of μ_A , namely composition of μ_A with the projection on the space $\bigoplus_{j=1,2} H^0(C_j, \omega_{C_j}^{\otimes 3}(2D_j))$. So, we can decompose the map $\nu_A = \nu_1 \oplus \nu_2$ where

$$\nu_j : \Lambda^2 H^0(C, \omega_C \otimes A) \rightarrow H^0(C_j, \omega_{C_j}^{\otimes 3}(2D_j)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2g-4)).$$

In the coordinate t , the embeddings defining the two components of C are $\phi_j(t, 1) = [\alpha_{1,j}(t) \dots \alpha_{g-1,j}(t)]$, where the components $\alpha_{i,j}$ are defined as in (1). Let $\sigma_1 \dots \sigma_{g-1}$ be the basis of $H^0(\omega_C \otimes A)$ given by the coordinate hyperplane sections. We have

$$(10) \quad \nu_h(\sigma_i \wedge \sigma_j) = (\alpha_{i,h} \alpha'_{j,h} - \alpha_{j,h} \alpha'_{i,h})(dt)^3, \quad h = 1, 2$$

So, as an element of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2g-4))$ we can identify $\nu_h(\sigma_i \wedge \sigma_j)$ with the polynomial $\nu_{ij,h}$, for which we have the following expressions, if in (1) we choose $d_2 = 1$ and so $d_1 = -\frac{A_1}{A_2}$:

$$(11) \quad \begin{aligned} \nu_{ij,h}(t) &= (a_{i,h} - a_{j,h})t^2 \frac{M_h^2(t, 1)}{(t - a_{i,h})^2(t - a_{j,h})^2} && \text{for } i < j \leq \left\lfloor \frac{g}{2} \right\rfloor \\ \nu_{ij,h}(t) &= \frac{(a_{i,h} - a_{j,h})a_{i,h}a_{j,h}}{A_2^2} \frac{M_h^2(t, 1)}{(t - a_{i,h})^2(t - a_{j,h})^2} && \text{for } \left\lfloor \frac{g}{2} \right\rfloor + 1 \leq i < j \\ \nu_{ij,h}(t) &= \frac{(-1)^h a_{j,h}}{A_2} (t^2 - 2a_{i,h}t + a_{j,h}a_{i,h}) \frac{M_h^2(t, 1)}{(t - a_{i,h})^2(t - a_{j,h})^2} && \text{for } i \leq \left\lfloor \frac{g}{2} \right\rfloor < j \end{aligned}$$

We have the following commutative diagram with exact rows

$$(12) \quad \begin{array}{ccccccc} 0 & \longrightarrow & T & \xrightarrow[\quad i \quad]{\quad s \quad} & H^0(\Omega_C^1 \otimes \omega_C^{\otimes 2}) & \xrightarrow{p=p_1 \oplus p_2} & \bigoplus_{j=1,2} H^0(C_j, \omega_{C_j}^{\otimes 3}(2D_j)) \longrightarrow 0 \\ & & \uparrow t & & \uparrow \mu_A & \nearrow \nu_A = \nu_1 \oplus \nu_2 & \\ 0 & \longrightarrow & \text{Ker}(\nu_A) & \longrightarrow & \Lambda^2 H^0(C, \omega_C \otimes A) & & \end{array}$$

where, if the local expression of C in a neighborhood of the node P_h is $xy = 0$ and $\sigma \in H^0(\Omega_C^1 \otimes \omega_C^{\otimes 2})$, the h -th component of $s(\sigma)$ at the node P_h , $h = 1, \dots, g$ is given by the coefficient of the torsion term ydx in the local expression of $s(\sigma)$ at the node P_h . The map t is the restriction to the kernel of ν_A of the map $\tau = s \circ \mu_A : \Lambda^2 H^0(C, \omega_C \otimes A) \rightarrow T$, whose component at the node P_h , $h = 1, \dots, g$ can be easily computed as follows:

$$(13) \quad \tau(\sigma_i \wedge \sigma_j)_h = \alpha'_{j,1}(a_{h,1})\alpha'_{i,2}(a_{h,2}) - \alpha'_{i,1}(a_{h,1})\alpha'_{j,2}(a_{h,2}).$$

Similarly, for the component at the node P_{g+1} , one has:

$$(14) \quad \tau(\sigma_i \wedge \sigma_j)_{g+1} = g'_{j,1}(0)g'_{i,2}(0) - g'_{i,1}(0)g'_{j,2}(0),$$

where $g_{i,r}(u)$ is the i -th component of $\phi_r(1, u)$, $r = 1, 2$ (see (1) for the explicit expression). So we have $\mu_A = \nu_A \oplus \tau$, and in the next section we will prove by induction on the genus that it is surjective for the general Prym-canonical binary curve.

4. SURJECTIVITY

Let $C \subset \mathbb{P}^{g-2}$ be a Prym-canonical binary curve embedded by $\omega_C \otimes A$, with $A^{\otimes 2} \cong \mathcal{O}_C$, as in 3.1 and set $k := \lfloor \frac{g}{2} \rfloor$ and denote by \tilde{C}_r the partial normalization of C at the node P_r with $r = k$ if $g = 2k$,

$r = k + 1$ if $g = 2k + 1$ and let p_1, p_2 the two points of \tilde{C}_r over P_r . We make this choice of the node in order to obtain the Prym-canonical model for the curve \tilde{C}_r . In fact, observe that in this way, for a general choice of the $a_{i,j}$'s, the projection from P_r sends the curve C to the Prym-canonical model of \tilde{C}_r in \mathbb{P}^{g-3} given by the line bundle $\omega_{\tilde{C}_r} \otimes A'_r$ where A'_r corresponds to the point $(h'_1, \dots, h'_{g-1}, 1) \in \mathbb{C}^{*g}/\mathbb{C}^*$, with $h'_i = 1$ for $i \leq [\frac{g-1}{2}]$, $h'_i = -1$ for $i = [\frac{g-1}{2}] + 1, \dots, g-1$, as described in section 2. In fact $[\tilde{C}_r, A'_r]$ is parametrized by $a'_{i,j} = a_{i,j}$ for $i \leq r-1, j = 1, 2$, $a'_{i,j} = a_{i+1,j}$ for $i \geq r, j = 1, 2$. So if we set $d'_j := \frac{d_j}{a_{r,j}}$, $j = 1, 2$, we clearly have a pair $[\tilde{C}_r, A'_r]$ as in (1). For simplicity let us choose as above $d_2 = 1$, so $d_1 = -\frac{A_1}{A_2}$, hence $d'_2 := \frac{1}{a_{r,2}}, d'_1 := -\frac{A_1}{A_2 a_{r,1}}$.

For the inductive step, we will use the same strategy used in [5] and [9]. Looking at the torsion and non-torsion part of the map μ_A separately, we have the following commutative diagrams with horizontal exact sequences

$$(15) \quad \begin{array}{ccccc} 0 \longrightarrow \bigoplus_{i=1,2} H^0(C_i, \omega_{C_i}^{\otimes 3}(2\tilde{D}_i)) & \longrightarrow & \bigoplus_{i=1,2} H^0(C_i, \omega_{C_i}^{\otimes 3}(2D_i)) & \longrightarrow & \bigoplus_{i=1,2} \mathcal{O}_{2p_i} \\ & \uparrow \tilde{\nu}_A & \uparrow \nu_A & \nearrow \chi & \\ 0 \longrightarrow \bigwedge^2 H^0(\tilde{C}_r, \omega_{\tilde{C}_r} \otimes A'_r) & \longrightarrow & \bigwedge^2 H^0(C, \omega_C \otimes A) & & \end{array}$$

$$(16) \quad \begin{array}{ccccc} 0 \longrightarrow \tilde{T} & \longrightarrow & T & \longrightarrow & T_P \\ & \uparrow \tilde{\tau} & \uparrow \tau & \nearrow \tau_P & \\ 0 \longrightarrow \bigwedge^2 H^0(\tilde{C}_r, \omega_{\tilde{C}_r} \otimes A'_r) & \longrightarrow & \bigwedge^2 H^0(C, \omega_C \otimes A) & & \end{array}$$

where $P = P_r$, D_i is the divisor of nodes of C on C_i and $\tilde{D}_i = D_i - p_i$ and ν_A, τ and $\tilde{\nu}_A, \tilde{\tau}$ are the maps defined in the previous section for C and \tilde{C}_r . Hence, if the maps $\tilde{\nu}_A \oplus \tilde{\tau}$ and $\chi \oplus \tau_P$ are surjective, then $\nu_A \oplus \tau = \mu_A$ is also surjective.

Recall that the map ν_A is $\nu_1 \oplus \nu_2$, where ν_1 and ν_2 are defined in (10), so we can write $\chi = \chi_1 \oplus \chi_2$, where χ_h is the composition of ν_i with the restriction to \mathcal{O}_{2p_i} , $i = 1, 2$. In local coordinates around P , $\chi_h(\sigma_i \wedge \sigma_j)$ is the pair $(\nu_{ij,h}(a_{r,h}), \nu'_{ij,h}(a_{r,h}))$ (where $r = k$ for $g = 2k$ and $r = k + 1$ for $g = 2k + 1$) corresponding to the evaluation of the polynomial $\nu_{ij,h}(t)$ of (10) and of its derivative at P . Instead τ_P is nothing else than the torsion part of the map μ_A in the node P (as in (13)). So we can prove the following

Theorem 4.1. *If $[C = C_1 \cup C_2, \omega_C \otimes A]$ is a Prym-canonical general binary curve of genus $g \geq 12$, then μ_A is surjective for C .*

Proof. The case $g = 12$ is done by computations with Maple (see section 5). For $g > 12$ we proceed by induction, as explained above, so we assume that the map $\tilde{\nu}_A \oplus \tilde{\tau}$ of the above diagrams is surjective, hence we only have to check the maximality of the rank of the following $5 \times \frac{1}{2}(g-1)(g-2)$ matrix, for every $g \geq 13$:

$$\left(\begin{array}{c} \nu_{ij,1}|_{t=a_{k,1}} \\ \frac{d}{dt}(\nu_{ij,1})|_{t=a_{k,1}} \\ \nu_{ij,2}|_{t=a_{k,2}} \\ \frac{d}{dt}(\nu_{ij,2})|_{t=a_{k,2}} \\ \tau_{ij}(P_k) \end{array} \right) \quad \boxed{\text{if } g = 2k} \quad \left(\begin{array}{c} \nu_{ij,1}|_{t=a_{k+1,1}} \\ \frac{d}{dt}(\nu_{ij,1})|_{t=a_{k+1,1}} \\ \nu_{ij,2}|_{t=a_{k+1,2}} \\ \frac{d}{dt}(\nu_{ij,2})|_{t=a_{k+1,2}} \\ \tau_{ij}(P_{k+1}) \end{array} \right) \quad \boxed{\text{if } g = 2k+1}$$

where $1 \leq i < j \leq g-1$ are the columns indexes.

Let us start with the even case $g = 2k$. We will see that the 5×5 submatrix with column indexes $(i, j) = (1, k), (2, k), (k, g-2), (k, g-1), (k-1, k+1)$ is invertible for some appropriate choices of the parameters $a_{\ell, \epsilon}$, $\epsilon = 1, 2$.

For $\boxed{i = 1, 2; j = g - 2, g - 1; \epsilon = 1, 2}$, taking $\boxed{\delta_\epsilon = (-1)^{\epsilon+1}}$, we have (see equations (11))

$$\begin{aligned} N_{i,\epsilon} &:= \nu_{ik,\epsilon}|_{t=a_{k,\epsilon}} = a_{k,\epsilon}^2 (a_{i,\epsilon} - a_{k,\epsilon}) \prod_{\ell \neq i,k} (a_{k,\epsilon} - a_{\ell,\epsilon})^2 \\ N'_{i,\epsilon} &:= \frac{d(\nu_{ik,\epsilon})}{dt}|_{t=a_{k,\epsilon}} = 2a_{k,\epsilon} (a_{i,\epsilon} - a_{k,\epsilon}) \prod_{\ell \neq i,k} (a_{k,\epsilon} - a_{\ell,\epsilon}) \left(\prod_{\ell \neq i,k} (a_{k,\epsilon} - a_{\ell,\epsilon}) + a_{k,\epsilon} \sum_{h \neq i,k} \prod_{\ell \neq i,k,h} (a_{k,\epsilon} - a_{\ell,\epsilon}) \right) \\ N_{j,\epsilon} &:= \nu_{kj,\epsilon}|_{t=a_{k,\epsilon}} = \delta_\epsilon \frac{a_{k,\epsilon} a_{j,\epsilon}}{A_2} (a_{k,\epsilon} - a_{j,\epsilon}) \prod_{\ell \neq k,j} (a_{k,\epsilon} - a_{\ell,\epsilon})^2 \\ N'_{j,\epsilon} &:= \frac{d(\nu_{kj,\epsilon})}{dt}|_{t=a_{k,\epsilon}} = \delta_\epsilon \frac{2a_{k,\epsilon} a_{j,\epsilon}}{A_2} (a_{k,\epsilon} - a_{j,\epsilon}) \prod_{\ell \neq k,j} (a_{k,\epsilon} - a_{\ell,\epsilon}) \cdot \sum_{h \neq k,j} \prod_{\ell \neq k,j,h} (a_{k,\epsilon} - a_{\ell,\epsilon}), \end{aligned}$$

while if $i, j \neq k$, we have $\nu_{ij,\epsilon}|_{t=a_{k,\epsilon}} = 0$ and $\frac{d(\nu_{ij,\epsilon})}{dt}|_{t=a_{k,\epsilon}} = 0$, hence every entry of our matrix with $i, j \neq k$ is zero. So, it's sufficient to show that, for an appropriate choice of the parameters $a_{i,\epsilon}$, we have $\tau_{k-1,k+1}(P_k) \neq 0$ and the following 4×4 matrix is invertible:

$$(17) \quad \begin{pmatrix} N_{1,1} & N_{2,1} & N_{g-2,1} & N_{g-1,1} \\ N'_{1,1} & N'_{2,1} & N'_{g-2,1} & N'_{g-1,1} \\ N_{1,2} & N_{2,2} & N_{g-2,2} & N_{g-1,2} \\ N'_{1,2} & N'_{2,2} & N'_{g-2,2} & N'_{g-1,2} \end{pmatrix}$$

First of all, perform the following simplifications:

- Multiply the last two columns by A_2 .
- Divide the first row by $a_{k,1} \prod_{\substack{\ell \neq 1,2,k \\ \ell \neq g-2,g-1}} (a_{k,1} - a_{\ell,1})^2 \cdot (a_{k,1} - a_{1,1})(a_{k,1} - a_{2,1})(a_{k,1} - a_{g-2,1})(a_{k,1} - a_{g-1,1})$ and similarly for the third row.
- Divide the second row by $2a_{k,1} \prod_{\ell \neq k} (a_{k,1} - a_{\ell,1})$ and similarly for the fourth row.

We obtain

$$\begin{aligned} N_{i,\epsilon} &= -a_{k,\epsilon} \prod_{\substack{\ell=1,2,g-2,g-1 \\ \ell \neq i}} (a_{k,\epsilon} - a_{\ell,\epsilon}) \\ N'_{i,\epsilon} &= - \left(\prod_{\ell \neq i,k} (a_{k,\epsilon} - a_{\ell,\epsilon}) + a_{k,\epsilon} \sum_{h \neq i,k} \prod_{\ell \neq i,k,h} (a_{k,\epsilon} - a_{\ell,\epsilon}) \right) \\ N_{j,\epsilon} &= \delta_\epsilon a_{j,\epsilon} \prod_{\substack{\ell=1,2,g-2,g-1 \\ \ell \neq j}} (a_{k,\epsilon} - a_{\ell,\epsilon}) \\ N'_{j,\epsilon} &= \delta_\epsilon a_{j,\epsilon} \sum_{h \neq k,j} \prod_{\ell \neq k,j,h} (a_{k,\epsilon} - a_{\ell,\epsilon}) \end{aligned}$$

Now, let $p_\epsilon = \prod_{\substack{\ell \neq 1,2,k \\ \ell \neq g-2,g-1}} (a_{k,\epsilon} - a_{\ell,\epsilon})$, $p_{h,\epsilon} = \frac{p_\epsilon}{(a_{k,\epsilon} - a_{h,\epsilon})}$ and $\hat{a}_{\ell,\epsilon} = (a_{k,\epsilon} - a_{\ell,\epsilon})$, for all $\ell \neq k$, so that we can write

$$\begin{aligned} N'_{i,\epsilon} &= - \left(p_\epsilon \left(\prod_{\substack{\ell=1,2,g-2,g-1 \\ \ell \neq i}} \hat{a}_{\ell,\epsilon} \right) + a_{k,\epsilon} \left(\prod_{\substack{\ell=1,2,g-2,g-1 \\ \ell \neq i}} \hat{a}_{\ell,\epsilon} \right) \left(\sum_{\substack{h \neq 1,2,k \\ g-2,g-1}} p_{h,\epsilon} \right) + a_{k,\epsilon} p_\epsilon \left(\sum_{\substack{n=1,2,g-2,g-1 \\ n \neq i}} \frac{\prod_{\substack{\ell=1,2,g-2,g-1 \\ \ell \neq i}} \hat{a}_{\ell,\epsilon}}{\hat{a}_{n,\epsilon}} \right) \right) \\ N'_{j,\epsilon} &= \delta_\epsilon \left(a_{j,\epsilon} \left(\prod_{\substack{\ell=1,2,g-2,g-1 \\ \ell \neq j}} \hat{a}_{\ell,\epsilon} \right) \left(\sum_{\substack{h \neq 1,2,k \\ g-2,g-1}} p_{h,\epsilon} \right) + a_{j,\epsilon} p_\epsilon \left(\sum_{\substack{n=1,2,g-2,g-1 \\ n \neq j}} \frac{\prod_{\substack{\ell=1,2,g-2,g-1 \\ \ell \neq j}} \hat{a}_{\ell,\epsilon}}{\hat{a}_{n,\epsilon}} \right) \right) \end{aligned}$$

Now we choose the following values for the parameters:

$$(18) \quad \boxed{a_{i,1} = i \cdot a, \quad a \neq 0, 1 \quad a_{i,2} = i}$$

So, we obtain $\sum_{\substack{h \neq 1, 2, k \\ g-2, g-1}} p_{h, \epsilon} = 0$ for $\epsilon = 1, 2$ and $p_1 = a^{g-6} p_2 = a^{g-6} (-1)^{k-3} ((k-3)!)^2$. Dividing the first row by a^4 , the second row by $a^3 p_1$, the fourth row by p_2 , and the first and third row by $(k-1)(k-2)$, we obtain the matrix

$$\begin{pmatrix} -k(k-2) & -k(k-1) & -2(k-1)^2 & -(2k-1)(k-2) \\ (k-2)^2 & 2(k-1)^2 & -2(k-1)^3 & -(2k-1)(k-2)^2 \\ -k(k-2) & -k(k-1) & 2(k-1)^2 & (2k-1)(k-2) \\ (k-2)^2 & 2(k-1)^2 & 2(k-1)^3 & (2k-1)(k-2)^2 \end{pmatrix}$$

that is invertible. Finally we can check that with the parameters (18) we have

$$\tau_{k-1, k+1}(P_k) = \frac{2k(k+1)a^{g-2}}{A_2} \prod_{\ell \neq k-1, k+1, k} (k-\ell)^2 \neq 0$$

The computations and strategy for the odd case $g = 2k+1$ are similar. We will prove that with the same choice for the $a_{i, \epsilon}$'s as in the even case, the 5×5 matrix with column indexes $(2, k+1), (3, k+1), (k+1, g-2), (k+1, g-1), (k-1, k+2)$ is invertible.

For the indexes $(i, j) = (2, k+1), (3, k+1), (k+1, g-2), (k+1, g-1)$ and $\delta_\epsilon = (-1)^{\epsilon+1}$, we have

$$\begin{aligned} N_{i, \epsilon} &= \nu_{i, k+1, \epsilon} |_{t=a_{k+1, \epsilon}} = -\delta_\epsilon \prod_{\ell \neq i, k+1} (a_{k+1, \epsilon} - a_{\ell, \epsilon})^2 \cdot \frac{(a_{k+1, \epsilon}^2 - a_{i, \epsilon} a_{k+1, \epsilon}) a_{k+1, \epsilon}}{A_2} \\ N'_{i, \epsilon} &= \frac{d(\nu_{i, k+1, \epsilon})}{dt} |_{t=a_{k+1, \epsilon}} = -\delta_\epsilon \frac{2a_{k+1, \epsilon}(a_{k+1, \epsilon} - a_{i, \epsilon}) \prod_{\ell \neq i, k+1} (a_{k+1, \epsilon} - a_{\ell, \epsilon})}{A_2} \cdot \left[\prod_{\ell \neq i, k+1} (a_{k+1, \epsilon} - a_{\ell, \epsilon}) + \right. \\ &\quad \left. + a_{k+1, \epsilon} \sum_{h \neq i, k+1} \prod_{\ell \neq i, k+1, h} (a_{k+1, \epsilon} - a_{\ell, \epsilon}) \right] \\ N_{j, \epsilon} &= \nu_{j+1, j, \epsilon} |_{t=a_{k+1, \epsilon}} = \prod_{\ell \neq j, k+1} (a_{k+1, \epsilon} - a_{\ell, \epsilon})^2 \cdot \frac{(a_{k+1, \epsilon} - a_{j, \epsilon}) a_{k+1, \epsilon} a_{j, \epsilon}}{A_2^2} \\ N'_{j, \epsilon} &= \frac{d(\nu_{j+1, j, \epsilon})}{dt} |_{t=a_{k+1, \epsilon}} = 2 \cdot \prod_{\ell \neq k+1, j} (a_{k+1, \epsilon} - a_{\ell, \epsilon}) \cdot \sum_{h \neq k+1, j} \prod_{\ell \neq k+1, j, h} (a_{k+1, \epsilon} - a_{\ell, \epsilon}) \cdot \frac{(a_{k+1, \epsilon} - a_{j, \epsilon}) a_{k+1, \epsilon} a_{j, \epsilon}}{A_2^2} \end{aligned}$$

Making analogous simplifications as in the even case, and choosing as above the parameters $a_{i, 1} = ia$, $a \neq 0, 1$, $a_{i, 2} = i$, we obtain the matrix

$$\begin{pmatrix} -(k+1)(k-1) & -(k+1)(k-2) & -(k-2) & -(k-1) \\ -(k^2 - 2k - 1) & -(k^2 - 4k - 2) & -(2k-2) & -(2k-1) \\ (k+1)(k-1) & (k+1)(k-2) & -(k-2) & -(k-1) \\ (k^2 - 2k - 1) & (k^2 - 4k - 2) & -(2k-2) & -(2k-1) \end{pmatrix}$$

that is an invertible. Finally we observe that if $i, j \neq k+1$, we have $\nu_{ij, \epsilon} |_{t=a_{k+1, \epsilon}} = 0$ and $\frac{d(\nu_{ij, \epsilon})}{dt} |_{t=a_{k+1, \epsilon}} = 0$, and

$$\tau_{k-1, k+2}(P_{k+1}) = \frac{-4(k+1)(k+2)a^{g-2}}{A_2} \prod_{\ell \neq k-1, k+2, k+1} (k+1-\ell)^2 \neq 0.$$

□

5. MAPLE SCRIPTS FOR COMPUTATIONS AND INJECTIVITY FOR LOW GENUS

In this section we want to show the following

Theorem 5.1. *If $[C = C_1 \cup C_2, \omega_C \otimes A]$ is a Prym-canonical general binary curve of genus $3 \leq g \leq 12$, then μ_A is injective for C .*

Proof. For the cases $g = 3, 4$, assume that C is a smooth curve and let $\omega_C \otimes A$ be a Prym-canonical line bundle, then, given a non zero decomposable vector $\sigma_1 \wedge \sigma_2 \in \Lambda^2 H^0(\omega_C \otimes A)$, we have $\mu_A(\sigma_1 \wedge \sigma_2) \neq 0$, as one can easily check, or find in [15], (1.4). This concludes the proof for genus 3 curves. Now, by a simple linear algebra remark (see e.g. [10], (2.4)), one sees that $\text{rank}(\mu_A) \geq \dim(G(2, h^0(\omega_C \otimes A))) + 1 = 2h^0(\omega_C \otimes A) - 3$, where $G(2, h^0(\omega_C \otimes A))$ is the Grassmannian of 2-dimensional subspaces of $H^0(\omega_C \otimes A)$. So for any curve of genus 4, we have $\text{rank}(\mu_A) \geq 2h^0(\omega_C \otimes A) - 3 = \dim(\Lambda^2 H^0(\omega_C \otimes A)) = 3$.

For $g \geq 5$ we can use the model of Prym-canonical binary curves presented in section 2. We will choose some random values for the parameters $a_{i,j}$ and check, making the computation with Maple, that there exists a Prym-canonical curve constructed like in section 2 for which the map μ_A has maximal rank. Observe that for $g = 12$ we have $\dim(\Lambda^2 H^0(\omega_C \otimes A)) = \dim(H^0(\Omega_C^1 \otimes \omega_C^{\otimes 2})) = 55$, so the maximality of the rank of μ_A for genus 12 is the basis of the induction for the proof of the surjectivity for $g \geq 12$, as explained in the previous section. For $4 < g < 12$ we obtain, by semicontinuity, the injectivity of μ_A for a general point of \mathcal{R}_g . The following is the Maple script used to construct the matrices of the torsion and non torsion part of μ_A in the way explained in section 3.3 and to check the rank for genus $4 \leq g \leq 12$:

```
(1) with(LinearAlgebra):
v:=[1,2,3,4,5,6,7,8,9,10,11]: w:=[326,-28,-875,-97,20,-651,-523,-306,369,-31,99]:
for g from 4 to 12 do
unassign(a): unassign(b): a[g]:=0: b[g]:=0: k:=floor(g/2):
(5)   for i from 1 to k do P[i]:=0 end do:
      for i from k+1 to g-1 do P[i]:=1 end do:
      A2:=mul(b[i],i=1..g-1):
      c[1]:=[seq(-P[i]*a[i]*A2,i=1..g-1)]: c[2]:=[seq(P[i]*b[i]/A2,i=1..g-1)]:
      for i from 1 to k do delta[i]:=1 end do:
(10)  for i from k+1 to g-1 do delta[i]:=0 end do:
      M[1]:=mul(t-a[i], i = 1..g-1): M[2] := mul(t-b[i], i = 1..g-1):
      for i from 1 to g-1 do
        alpha[i,1]:=M[1]*(delta[i]*t-c[1][i])/(t-a[i]);
        alpha[i,2]:=M[2]*(delta[i]*t-c[2][i])/(t-b[i])
      end do:
(15)  for i from 1 to g-1 do for h from 1 to 2 do
      alphaprimo[i,h]:=diff(alpha[i,h],t);
    end do: end do:
    for i from 1 to g-2 do for j from i+1 to g-1 do for h from 1 to 2 do
      R[h][i,j]:=alpha[i,h]*alphaprimo[j,h] - alpha[j,h]*alphaprimo[i,h];
(20)  end do: end do: end do:
      MM[1]:=mul(1-a[i]*u, i = 1..g-1): MM[2] := mul(1-b[i]*u, i = 1..g-1):
      for i from 1 to g-1 do
        tauGp1a[i]:=diff(MM[1]*(delta[i]-c[1][i]*u)/(1-a[i]*u),u);
        tauGp1b[i]:=diff(MM[2]*(delta[i]-c[2][i]*u)/(1-b[i]*u),u)
(25)  end do:
      for i from 1 to g-2 do for j from i+1 to g-1 do for h from 1 to g do
        tia:=eval(alphaprimo[i,1],t=a[h]); tja:=eval(alphaprimo[j,1],t=a[h]);
        tjb:=eval(alphaprimo[j,2],t=b[h]); tib:=eval(alphaprimo[i,2],t=b[h]);
        tau[h][i,j] := (tja)*(tib) - (tia)*(tjb)
(30)  end do: end do: end do:
      for i from 1 to g-2 do for j from i+1 to g-1 do
        seqnu:=seq( seq(coeff(R[h][i,j],t,n), n=0..2*g-4), h=1..2 );
        tGp1ia:=eval(tauGp1a[i],u=0); tGp1jb:=eval(tauGp1b[j],u=0);
        tGp1ja:=eval(tauGp1a[j],u=0); tGp1ib:=eval(tauGp1b[i],u=0);
(35)  seqtau:=seq(tau[h][i,j], h=1..g);
        riga[i,j]:=[ seqnu, seqtau, (tGp1ja)*(tGp1ib)-(tGp1ia)*(tGp1jb) ]
      end do: end do:
      ttt[g]:=[ seq(seq(riga[i,j], j=i+1..g-1), i=1..g-2) ]:
      T[g]:=convert(ttt[g], Matrix):
```



```

(40)  mappamu[g]:=eval( eval(T[g], a=v[1..g-1]), b=w[1..g-1] ):
      rango[g]:=Rank(mappamu[g]):
end do:
for g from 4 to 12 do T[g]; rango[g]; end do;

```

We will briefly explain the code. At the second line we take the vectors with the choice of the values that we will use for parameters. From line 4 to line 14 we calculate the functions $\alpha_{i,j}$ which represent the embeddings $\phi_j(t, 1)$. Note that the sequence memorized in the variable **a** is the sequence of the parameters $a_{i,1}$, $i = 1, \dots, g-1$, while the parameters $a_{i,2}$ are memorized in the variable **b**. From line 15 to 20, we calculate $\alpha'_{i,j}$ and the polynomials $\nu_{ij,h}$ (i.e. $R[h][i,j]$) as in equation (10). In lines 21–25 we calculate the functions $g'_{i,r}$ (called **tauGp1a** and **tauGp1b** in the code), i.e. the derivative of the $(g+1)$ -coordinate of $\phi_r(1, u)$, $r = 1, 2$, needed for the torsion part in the last node (see equation (14)). In lines 24–30 we calculate the h -component, for $h = 1, \dots, g$, of the torsion for every $\sigma_i \wedge \sigma_j$ (as in equation (13)). Then we take the coefficients of the polynomials $\nu_{ij,h}$ (line 32) and assemble the matrix $\begin{pmatrix} \nu_A \\ \tau \end{pmatrix}$ (line 39, for convenience in the code, we built the transpose of this matrix). Finally, we replace the values in the parameters (line 40) and we calculate the rank of the matrix, which turns out to be maximal for $g = 4, \dots, 12$. \square

6. THE CLASS

In the previous sections we have proved by semicontinuity that the first Gaussian map $\mu_A : \Lambda^2 H^0(\omega_C \otimes A) \rightarrow H^0(\omega_C^{\otimes 2} \otimes \Omega_C^1)$ has maximal rank for the general pair $[C, A]$ in \mathcal{R}_{12} . Notice that for $g = 12$, $\dim(\Lambda^2 H^0(\omega_C \otimes A)) = \dim(H^0(\omega_C^{\otimes 2} \otimes \Omega_C^1)) = 55$. Consider the locus $\mathcal{D} = \{[C, A] \in \mathcal{R}_{12} \mid rk(\mu_A) < 55\}$. We have proved that $\mathcal{D} \neq \mathcal{R}_{12}$, hence, if it is not empty, it is an effective divisor in \mathcal{R}_{12} . Let $\pi : \overline{\mathcal{R}}_g \rightarrow \overline{\mathcal{M}}_g$ be the finite map which extends the forgetful map $\mathcal{R}_g \rightarrow \mathcal{M}_g$ (see [12] Section 1). The partial compactification $\tilde{\mathcal{R}}_g$ of \mathcal{R}_g introduced in [12] Section 1 is the inverse image $\pi^{-1}(\tilde{\mathcal{M}}_g)$, where $\tilde{\mathcal{M}}_g := \mathcal{M}_g \cup \tilde{\Delta}_0$ and $\tilde{\Delta}_0$ is the locus of one-nodal irreducible curves. Denote by $\psi : \mathcal{X} \rightarrow \tilde{\mathcal{R}}_g$ the universal family and by $\mathcal{P} \in Pic(\mathcal{X})$ the corresponding Prym bundle as in [12] 1.1. Assume $g = 12$, then if $\tilde{\mathcal{D}}$ is the closure of \mathcal{D} in $\tilde{\mathcal{R}}_{12}$, $\tilde{\mathcal{D}}$ is the degeneracy locus of the map

$$\Lambda^2(\psi_*(\omega_\psi \otimes \mathcal{P})) \rightarrow \psi_*((\omega_\psi \otimes \mathcal{P})^{\otimes 2} \otimes \Omega_\psi^1) \cong \psi_*(\omega_\psi^{\otimes 3} \otimes \mathcal{P}^{\otimes 2} \otimes \mathcal{I}_Z)$$

of (6), where $Z = Sing(\psi)$, $\Omega_\psi^1 \cong \omega_\psi \otimes \mathcal{I}_Z$. Denote by $\mathcal{F}_i := \psi_*(\omega_\psi^{\otimes i} \otimes \mathcal{P}^{\otimes i})$. Using Grothendieck-Riemann-Roch and Proposition 1.6 of [12] one computes as in Proposition 1.7 of [12]

$$c_1(\mathcal{F}_i) = \frac{i(i-1)}{2}(12\lambda - \delta'_0 - \delta''_0 - 2\delta_0^{ram}) + \lambda - \frac{i^2}{4}\delta_0^{ram},$$

where λ is the pullback of the Hodge class $\lambda \in \overline{\mathcal{M}}_g$ and δ'_0, δ''_0 , and δ_0^{ram} are the boundary classes defined in [12] section 1. So we have $c_1(\mathcal{F}_1) = \lambda - \frac{\delta_0^{ram}}{4}$, $c_1(\Lambda^2(\psi_*(\omega_\psi \otimes \mathcal{P}))) = 10\lambda - \frac{5}{2}\delta_0^{ram}$.

Notice that by Grothendieck-Riemann-Roch we have

$$\begin{aligned} c_1(\psi_*(\omega_\psi^{\otimes 3} \otimes \mathcal{P}^{\otimes 2} \otimes \mathcal{I}_Z)) &= \psi_*[(1 + c_1(\omega_\psi^{\otimes 3} \otimes \mathcal{P}^{\otimes 2}) + \frac{1}{2}c_1^2(\omega_\psi^{\otimes 3} \otimes \mathcal{P}^{\otimes 2}) - [Z]) \cdot (1 - \frac{c_1(\omega_\psi)}{2} + \frac{c_1(\omega_\psi)^2 + [Z]}{12})]_2 \\ &= 37\lambda - 4(\delta'_0 + \delta''_0) - 9\delta_0^{ram}, \end{aligned}$$

since $\psi_*(c_1(\omega_\psi) \cdot \mathcal{P}) = 0$, $\psi_*(c_1(\mathcal{P})^2) = -\delta_0^{ram}/2$, by Proposition 1.6 of [12] and by Mumford's formula, $\psi_*(c_1(\omega_\psi)^2) = 12\lambda - \psi_*([Z])$ and $\psi_*[Z] = \delta'_0 + \delta''_0 + 2\delta_0^{ram}$ ([12], 1.1). So, finally we have

$$\begin{aligned} c_1(\tilde{\mathcal{D}}) &= c_1(\psi_*(\omega_\psi^{\otimes 3} \otimes \mathcal{P}^{\otimes 2} \otimes \mathcal{I}_Z)) \cdot rk(\Lambda^2(\psi_*(\omega_\psi \otimes \mathcal{P})) - c_1(\Lambda^2(\psi_*(\omega_\psi \otimes \mathcal{P}))) \cdot rk(\psi_*(\omega_\psi^{\otimes 3} \otimes \mathcal{P}^{\otimes 2} \otimes \mathcal{I}_Z))) = \\ &= 55(27\lambda - 4(\delta'_0 + \delta''_0) - \frac{13}{2}\delta_0^{ram}), \end{aligned}$$

and $c_1(\mathcal{D}) = 1485\lambda$, hence \mathcal{D} is an effective divisor in \mathcal{R}_{12} , $\tilde{\mathcal{D}}$ is an effective divisor in $\tilde{\mathcal{R}}_{12}$ and if we denote by $\overline{\mathcal{D}}$ the closure of \mathcal{D} in $\overline{\mathcal{R}}_{12}$, we have computed

$$(19) \quad c_1(\overline{\mathcal{D}}) = 55(27\lambda - 4(\delta'_0 + \delta''_0) - \frac{13}{2}\delta_0^{ram} - \dots)$$

In fact, since the partial compactification $\tilde{\mathcal{R}}_g \subset \overline{\mathcal{R}}_g$ has the property that $\pi^{-1}(\mathcal{M}_g \cup \Delta_0) - \tilde{\mathcal{R}}_g$ has codimension ≥ 2 , the expression (19) computes the coefficients of $\lambda, \delta'_0, \delta''_0, \delta_0^{ram}$ in $c_1(\overline{\mathcal{D}})$.

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